ON HAMILTONICITY OF {CLAW, NET}-FREE GRAPHS

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Abstract

An *st-path* is a path with the end-vertices s and t. An *s-path* is a path with an end-vertex s. The results of this paper include necessary and sufficient conditions for a {claw, net}-free graph G with $s, t \in V(G)$ and $e \in E(G)$ to have (1) a Hamiltonian s-path, (2) a Hamiltonian st-path, (3) a Hamiltonian s- and st-paths containing e when G has connectivity one, and (4) a Hamiltonian cycle containing e when G is 2-connected. These results imply that a connected {claw, net}-free graph has a Hamiltonian path and a 2-connected {claw, net}-free graph has a Hamiltonian cycle [3]. Our proofs of (1)-(4) are shorter than the proofs of their corollaries in [3], and provide polynomial-time algorithms for solving the corresponding Hamiltonicity problems.

Keywords: claw, net, graph, {claw, net}-free graph, Hamiltonian path, Hamiltonian cycle, polynomial-time algorithm.

1 Introduction

We consider simple undirected graphs. All notions on graphs that are not defined here can be found in [2, 12].

A graph G is called *H-free* if G has no induced subgraph isomorphic to a graph H. A *claw* is a graph having exactly four vertices and exactly three edges that are incident to a common vertex. A claw can be drawn as the letter Y. A *net* is a graph obtained from a triangle by attaching to each vertex a new dangling edge.

There are many papers devoted to the study of Hamiltonicity of claw-free graphs, and, in particular, {claw, net}-free graphs (e.g. [1, 3, 4, 6, 7, 8, 10, 11]). The maximum independent vertex set problem for {claw, net}-free graphs was studied in [5]. In this paper we establish some new Hamiltonicity results on {claw, net}-free graphs.

An *st-path* is a path with the end-vertices s and t. An *s-path* is a path with an end-vertex s. Let G be a {claw, net}-free graph, $s, t \in V(G)$, $s \neq t$, and $e \in E(G)$. The results of this paper include necessary and sufficient conditions for G to have:

a Hamiltonian s-path (see **4.3** and **4.9** below),
a Hamiltonian st-path when G has connectivity one (see **4.3**),
a Hamiltonian st-path containing e if G has connectivity one (**4.6**),
a Hamiltonian s-path containing e when G has connectivity one (**4.7**), and
a Hamiltonian cycle containing e when G is 2-connected (**4.9**).

From the above mentioned results we have the following corollaries.

1.1 [3] (Corollary of **4.3**) *Every connected {claw, net}-free graph has a Hamiltonian path.*

1.2 [3] (Corollary of **4.9**) *Every 2-connected {claw, net}-free graph has a Hamiltonian cycle.*

Our proofs of **4.3** and **4.9** are shorter and more natural than the proofs of their corollaries **1.1** and **1.2** in [3]. They also provide polynomial time algorithms for solving the corresponding Hamiltonian problems for {claw, net}-free graphs. In [1] a linear time algorithm was given for finding a Hamiltonian path and a Hamiltonian cycle (if any exist) in a {claw, net}-free graph.

The known results on 3-connected {claw, net}-free graphs include the following.

1.3 [11] *A 3-connected {claw, net}-free graph has a Hamiltonian xy-path for every two distinct vertices x and y.*

1.4 [8] *Let G be a {claw, net}-free graph. If G is 3-connected, then every two non-adjacent edges in G belong to a Hamiltonian cycle. If G is 4-connected, then every two edges in G belong to a Hamiltonian cycle.*

1.5 [8] *Let G be a 3-connected {claw, net}-free graph, $e = uv \in E(G)$, and $s, t \in V(G)$, $s \neq t$. Then G has a Hamiltonian st-path containing e if and only if either $\{s,t\} \cap \{u,v\} = \emptyset$ or $\{s,t\} \setminus \{u,v\} = z \in V(G)$ and $G - \{z,u,v\}$ is connected.*

1.6 [8] *Let G be a k-connected {claw, net}-free graph, $k \geq 3$, L_1 and L_2 two disjoint paths in G, $|V(L_1)| + |V(L_2)| \leq k$, and x_1, x_2 the end-vertices of L_1, L_2, respectively. Then the following are equivalent:*

(c1) G has a Hamiltonian x_1x_2-path containing L_1 and L_2,

(c2) G has a Hamiltonian z_1z_2-path containing L_1 and L_2 for every end-vertices z_1, z_2 of L_1, L_2, respectively, and

(c3) $G - (L_1 \cup L_2)$ is connected.

1.7 [8] *Let G be a k-connected {claw, net}-free graph, $k \geq 2$, L a path in G, and $|V(L)| \leq k$. Then G has a Hamiltonian cycle containing L if and only if $G - L$ is connected.*

Obviously both **1.3** and **1.4** follow immediately from **1.5**. More results on Hamiltonicity of k-connected {claw, net}-free graphs can be found in [8].

The results of this paper form a part of a broader picture on Hamiltonicity of {claw, net}-free graphs and were presented at the Discrete Mathematics Seminar at the University of Puerto Rico in November 1999 (see also [8, 6]).

2 Main notions and notation

We consider undirected graphs with no loops and no parallel edges. We use the following notation: $V(G)$ and $E(G)$ are the sets of vertices and edges of a graph G, respectively, $v(G) = |V(G)|$ and $e(G) = |E(G)|$, AvB is the union of two graphs A and B having exactly one vertex v in common, and $AvB = Avu$ if B is an edge vu.

An *st-path* (*s-path*) is a path with the end-vertices s and t (an end-vertex s, respectively). If a and b are vertices of P, then aPb denote the subpath of P with the

end-vertices a and b. A path (a cycle) of G is called *Hamiltonian* if it contains each vertex of G. A Hamiltonian path of G is also called a *trace* of G. We introduce the term *track* of G for a Hamiltonian cycle of G.

Let $\kappa(G)$ denote the *vertex connectivity* of a graph G. A graph G is called *k-connected* if $\kappa(G) \ge k$.

Let H be a subgraph of G. We write simply $G - H$ instead of $G - V(H)$. A vertex x of H is called an *inner vertex of H* if x is adjacent to no vertices in $G - H$, and a *boundary vertex of H*, otherwise. An edge e of H is called an *inner edge of H* if e is incident to an inner vertex of H.

A *block* of G is either an isolated vertex or a maximal connected subgraph H of G such that $H - v$ is connected for every $v \in V(H)$. A block B of G is called an *end-block of G* if B has exactly one boundary vertex, and an *inner block*, otherwise.

3 The key lemma

First we observe the following.

3.1 *Let G be a graph. The following are equivalent:*

$(a1)$ *G has no induced subgraph isomorphic to a claw or a net and*

$(a2)$ *G has no connected induced subgraph with at least three end-blocks.*

Proof Obviously $(a2) \Rightarrow (a1)$. We prove $(a1) \Rightarrow (a2)$. If G is {claw, net}-free, then $G - x$ is also {claw, net}-free for every $x \in V(G)$. Clearly our claim is true if $v(G) = 1$. Let F be a counterexample with the minimum number of vertices. Then (1) every end-block has exactly one edge, (2) F has exactly three end-blocks, (3) if $x \in V(F)$ and $F - x$ is connected, then x is a leaf, and (4) F is not a claw and not a net. By (2) and (3), F is a tree or has exactly one cycle which is a triangle. In both cases by (4), F has a leaf z such that $F - z$ is a smaller counterexample, a contradiction. $\square$

The following lemma is useful for analyzing Hamiltonicity of {claw, net}-free graphs.

3.2 *Let G be a {claw, net}-free graph and $z \in V(G)$. Suppose that $G - z$ has an xy-trace P and there exists $e_z = zp \in E(G)$, and so G is connected and $p \in V(P)$. Let e_x and e_y be the end-edges of P. Then G has an ab-trace Q such that $\{a, b\} \subset \{x, y, z\}$, $e_z \in E(Q)$ and $\{e_x, e_y\} \cap E(Q) \ne \emptyset$.*

Proof (uses **3.1**). We define below a notion of a *good path* which is a special subpath of path P. Our goal is to show that if G has no required trace, then G has a good path and a maximal good path is a subpath of a longer good path in G, which is a contradiction.

By the assumption of our claim, $p \in V(P)$. Let $X = pPx = x_0x_1 \cdots x_{k-1}x_k$ and $Y = pPy = y_0y_1 \cdots y_t$, where $x_k = x$, $y_t = y$, and $x_0 = y_0 = p$. Let $M_{r,s} = x_rPy_s$, $\dot{M}_{r,s}$ denote the subgraph of G induced by $V(M_{r,s})$, and $\bar{M}_{r,s} = \dot{M}_{r,s} \cup \{x_rx_{r+1}, y_sy_{s+1}, zp\}$.

A subpath $M_{r,s}$ is called *good* if

(x1) $\dot{M}_{r,s}$ has a py_s-trace containing $x_{r-1}x_r$,

(y1) $\dot{M}_{r,s}$ has a px_r-trace containing $y_{s-1}y_s$,

(x2) if $x_r \neq x$, then for every $v \in V(M_{r,s}) \setminus x_r$, the graph $\dot{M}_{r,s} \cup \{x_r x_{r+1}, x_{r+1} v\}$ obtained from $\dot{M}_{r,s}$ by adding the edge $x_r x_{r+1}$ and a new edge $x_{r+1} v$ has a py_s-trace containing the path $x_r x_{r+1} v$,

(y2) if $y_s \neq y$, then for every $v \in V(M_{r,s}) \setminus y_s$, the graph $\dot{M}_{r,s} \cup \{y_s y_{s+1}, y_{s+1} v\}$ obtained from $\dot{M}_{r,s}$ by adding the edge $y_s y_{s+1}$ and a new edge $y_{s+1} v$ has a px_r-trace containing the path $y_s y_{s+1} v$, and

(z) for every $v \in V(M_{r,s}) \setminus p$, the graph $\dot{M}_{r,s} \cup \{zp, zv\}$ obtained from $\dot{M}_{r,s}$ by adding the edge zp and a new edge zv has an $x_r y_s$-trace (which clearly contains $e_z = zp$ and zv).

If $p \in \{x, y\}$ or $\{x_1 z, y_1 z\} \cap E(G) \neq \emptyset$, then clearly G has a required trace. Therefore let $p \notin \{x, y\}$ and $\{x_1 z, y_1 z\} \cap E(G) = \emptyset$. Since G has no induced claws, the claw in G with the edge set $\{px_1, py_1, pz\}$ is not induced, and therefore $x_1 y_1 \in E(G)$.

Clearly $\dot{M}_{1,1}$ is a triangle and $V(\dot{M}_{1,1}) = \{p, x_1, x_2\}$. Now it is easy to check that $M_{1,1}$ is a good path. Let $M_{r,s}$ be a maximal good path. Put $A = \{e_x, e_y, e_z\}$.

(**p1**) Suppose that $x_r = x$. By **(x1)**, $\dot{M}_{r,s}$ has a py_s-trace L containing $x_{r-1} x_r$. Then $zpLy_sPy$ is a yz-trace in G containing A. Similarly, if $y_s = y$, then G has an xz-trace containing A.

(**p2**) Now suppose that $x_r \neq x$ and $y_s \neq y$. Then the subgraph $\bar{M}_{r,s}$ of G has at least three end-blocks. Since G is {claw, net}-free, by **3.1**, there exists an edge ab in G such that $a \in \{x_{r+1}, y_{s+1}, z\}$ and $b \in V(\bar{M}_{r,s} - a)$.

(**p2.1**) Suppose that $a = z$ and $b \in V(M_{r,s})$. By **(z)**, $\bar{M}_{r,s} \cup zb$ has an $x_{r+1} y_{s+1}$-trace L containing e_z. Then $xPx_{r+1}Ly_{s+1}Py$ is an xy-trace in G containing A.

(**p2.2**) Suppose that $a = z$ and $b \in \{x_{r+1}, y_{s+1}\}$. By symmetry, we can assume that $b = x_{r+1}$. By **(x1)**, $\dot{M}_{r,s}$ has a py_s-trace L. Then $P' = xPx_{r+1}zpLy_sPy$ is an xy-trace in G. If $x \neq x_{r+1}$, then P' contains A. If $x = x_{r+1}$, then P' contains $A \setminus e_x$.

(**p2.3**) Now suppose that $a \in \{x_{r+1}, y_{s+1}\}$ and $b \neq z$. By symmetry, we can assume that $a = x_{r+1}$. Then $b \in V(M_{r,s} - x_r) \cup y_{s+1}$.

(**p2.3.1**) Suppose that $x_{r+1} = x$.

Suppose that $b \neq y_{s+1}$. By **(x2)**, $M_{r,s} \cup xb$ has a zy_s-trace L containing $e_x = x_r x_{r+1}$. Then $zpLy_s y_{s+1}Py$ is a yz-trace in G containing A.

Now suppose that $b = y_{s+1}$. By **(y1)**, $\dot{M}_{r,s}$ has a $\{p, x_r\}$-trace L. Then $P' = zpLx_r x_{r+1} y_{s+1}Py$ is a zy-trace in G. If $y_{s+1} \neq y$, then P' contains A. If $y_{s+1} = y$, then P' contains $A - e_y$.

(**p2.3.2**) Now suppose that $x_{r+1} \neq x$. Our goal is to show that

(c1) if $b \neq y_{s+1}$, then $M' = M_{r+1,s}$ is a good path and

(c2) if $b = y_{s+1}$ (i.e. $x_{r+1} y_{s+1} \in E(G)$), then $M' = M_{r+1,s+1}$ is a good path.

This will lead to a contradiction because $M_{r,s} \subset M'$, and therefore a good path $M_{r,s}$ will not be maximal. We recall that we consider the case when $x_r \neq x$ and $y_s \neq y$.

Case ($c1$). Suppose that $b \neq y_{s+1}$. We want to prove that $M_{r+1,s}$ is a good path.

(**p.x1**) Let us show that $M_{r+1,s}$ satisfies **(x1)**. By **(x2)** for $M_{r,s}$, the graph $\dot{M}_{r,s} \cup \{x_r x_{r+1}, x_{r+1} b\}$ has a py_s-trace L containing the path $x_r x_{r+1} b$. Then L is also a py_s-trace in $\dot{M}_{r+1,s}$ containing $x_r x_{r+1}$.

(**p.y1**) Let us show that $M_{r+1,s}$ satisfies **(y1)**. By **(y1)** for $M_{r,s}$, the graph $\dot{M}_{r,s}$ has

a px_r-trace L containing $y_{s-1}y_s$. Then pLx_rx_{r+1} is a px_{r+1}-trace in $\dot{M}_{r+1,s}$ containing $y_{s-1}y_s$.

(**p**.**x2**) Let us show that $M_{r+1,s}$ satisfies **(x2)**.

Consider graph $Q_v = \dot{M} \cup \{x_{r+1}x_{r+2}, x_{r+2}v\}$, where $v \in V(M_{r+1,s}) \setminus x_{r+1}$.

Suppose that $v \neq x_r$. By **(x2)** for $M_{r,s}$, graph $\dot{M}_{r,s} \cup \{x_rx_{r+1}, vx_{r+1}\}$ has a py_s-trace L containing the path $x_rx_{r+1}v$. Then $(L - vx_{r+1}) \cup (x_{r+1}x_{r+2}v)$ is a py_s-trace in Q_v containing path $x_{r+1}x_{r+2}v$.

Now suppose that $v = x_r$. By (**p**.**x1**), $M_{r+1,s}$ satisfies **(x1)**, i.e. graph $\dot{M}_{r+1,s}$ has a py_s-trace L containing x_rx_{r+1}. Then $(L - x_rx_{r+1}) \cup (x_{r+1}x_{r+2}x_r)$ is a py_s-trace containing path $x_{r+1}x_{r+2}v$.

(**p**.**y2**) Let us show that $M_{r+1,s}$ satisfies **(y2)**.

Consider graph $Q_v = \dot{M}_{r+1,s} \cup \{y_sy_{s+1}, vy_{s+1}\}$, where $v \in V(M_{r+1,s}) \setminus y_s$. By **(y2)** for $M_{r,s}$, graph $\dot{M}_{r,s} \cup \{y_sy_{s+1}, vy_{s+1}\}$ has a px_r-trace L containing path $y_sy_{s+1}v$. Then $x_{r+1}x_rLz$ is a $\{p, x_{r+1}\}$-trace in Q_v containing path $y_sy_{s+1}v$.

(**p**.**z**) Let us show that $M_{r+1,s}$ satisfies **(z)**.

Consider graph $Q_v = M_{r+1,s} \cup \{zp, zv\}$, where $v \in V(M_{r+1,s}) \setminus p$.

Suppose that $v \in V(M_{r,s}) \setminus p$. By **(z)** for $M_{r,s}$, graph $M_{rs} \cup \{zp, zv\}$ has an x_ry_s-trace L. Then $x_{r+1}x_rLy_s$ is an $x_{r+1}y_s$-trace in $M_{r+1,s} \cup \{zp, zv\}$.

Now suppose that $v = x_{r+1}$. By **(x1)** for $M_{r,s}$, graph $\dot{M}_{r,s}$ has a py_s-trace L. Then $x_{r+1}zpLy_s$ is an $x_{r+1}y_s$-trace in Q_v.

Case ($c2$). Now suppose that $b = y_{s+1}$. We want to prove that $M_{r+1,s+1}$ is a good path. By symmetry, it suffices to proof that $M_{r+1,s+1}$ satisfies **(x1)**, **(x2)**, and **(z)**. Let us proof **(x1)**. By **(y1)** for $M_{r,s}$, graph $\dot{M}_{r,s}$ has a px_r-trace L. Then $pLx_rx_{r+1}y_{s+1}$ is a py_{s+1}-trace in $\dot{M}_{r+1,s+1}$ containing x_rx_{r+1}. The proof of **(x2)** and **(z)** is similar to Case ($c1$). □

4 More on {claw, net}-free graph Hamiltonicity

Lemma **3.2** allows to give an easy proof of the following strengthening of **1.1**.

4.1 *Let G be a connected $\{$claw, net$\}$-free graph. Then*

($a1$) *G has a trace and*

($a2$) *if $sz \in E(G)$ and $G - z$ is connected, then sz belongs to a trace of G.*

Proof (uses **3.2**). We prove our claim by induction on $v(G)$. The claim holds if $v(G) = 1$. Since G is connected, there exists $z \in V(G)$ such that $G - z$ is also connected. Let $sz \in E(G)$. Since G is {claw, net}-free, clearly $G - z$ is also {claw, net}-free. Therefore by the induction hypothesis, $G - z$ has a trace. Then by **3.2**, G has a trace containing sz. □

Here is another strengthening of **1.1** for graphs of connectivity one.

4.2 *Let G be a connected $\{$claw, net$\}$-free graph, $G = AaHbB$, where A and B are end-blocks of G. Let $a' \in V(A - a)$, $b' \in V(B - b)$, and $a'x$ be an edge of A such that if $v(A) \geq 3$, then x is an inner vertex of an end-block of $G - a'$. Then*

($a1$) *there exists an $a'b'$-trace in G and, moreover,*

$(a2)$ *there exists an $a'b'$-trace in G containing edge $a'x$.*

Proof We prove our claim by induction on $v(G)$. If $v(G) = 3$, then our claim is obviously true.

(p1) Suppose that $v(A) \geq 3$. Then A is 2-connected. Let $A' = A - a'$ and $G' = G - a'$. Then $G' = A'aHbB$ and G' is connected. Since G is {claw, net}-free, G' is also {claw, net}-free. Since $v(G') < v(G)$, by the induction hypothesis, G' has an xb'-trace P. Then $a'xPb'$ is an $a'b'$-trace in G containing $a'x$.

(p2) Now suppose that $v(A) = 2$. Then $a'x = a'a$ and there is $b'z \in E(B)$ such that z is an inner vertex of an end-block in $G - b'$. Hence by the arguments, similar to those in **(p1)**, G has an $a'b'$-trace in G containing $a'x$ (as well as $b'z$). □

From **4.2** we have, in particular:

4.3 *Let G be a {claw, net}-free graph, $v(G) \geq 3$, $\kappa(G) = 1$, and $s, t \in V(G)$. Then G has an st-trace if and only if s and t are inner vertices of different end-blocks of G.*

From **4.1** and **4.2** it is easy to obtain the following stronger result.

4.4 *Let G be a connected {claw, net}-free graph having $k \geq 2$ blocks. Let A_j, $j \in \{1, 2\}$, be an end-block of G, a'_j the boundary vertex of A_j, $a_j \in A_j - a'_j$, and $\alpha_j \in E(A_j)$. Let B_i be an inner block of G and $\beta_i \in E(B_i)$. Let $U = \{\alpha_1, \alpha_2\} \cup \{\beta_i : i = 1, \dots, k-2\}$. Suppose that*

$(h1)$ *$\alpha_j = a_j x_j$ is such that if $v(A) \geq 3$, then x_j is an inner vertex of an end-block of $A_j - a'_j$, $j \in \{1, 2\}$, and*

$(h2)$ *β_i is an inner edge of B_i, if $v(B_i) \geq 3$, $i \in \{1, \dots, k-2\}$.*

Then G has an a_1a_2-trace containing U.

Proof (uses **4.1** and **4.2**). Since G is connected, for every end-block A_j of G there is an edge $a'_j p_j \in E(G) \setminus E(A_j)$. Similarly, for every inner block B_i of G there are edges $b_i q_j, b'_i q'_j \in E(G) \setminus E(B_i)$, where b_i and b'_i are the boundary vertices of B_i. Let $\bar{A}_j = A_j a'_j p_j$ and $\bar{B}_i = q_i b_i B_i b'_i q'_j$. Then all $\bar{A}_j$'s and $\bar{B}_i$'s are induced subgraphs of G and, therefore, are {claw, net}-free. By **4.1**, each $\bar{B}_i$ has a trace $q_i b_i Q_i b'_i q'_j$ containing β_i. By **4.2**, each $\bar{A}_j$ has a trace $a_j P_j a'_j p_j$ containing α_j. Then $P_1 \cup Q_1 \dots Q_{k-2} \cup P_2$ is an a_1a_2-trace containing U. □

Let $\mathcal{L}$ denote the set of 4-tuples (G, s, t, uv) such that G is a graph, $\{s, t\} \subseteq V(G)$, $s \neq t$, $uv \in E(G)$, and either (1) $\{s, t\}$ does not meet one of the components of $G - \{u, v\}$ or (2) $\{s, t\} \cap \{u, v\} \neq \emptyset$, say $t = u$, and either $G - \{s, v\}$ is not connected and the component containing t has at least two vertices or there is $x \in V(G - \{u, v\})$ such that $\{s, v\}$ avoids one of the components of $G - \{t, x\}$.

Obviously, if G has an st-trace containing uv, then $(G, s, t, uv) \notin \mathcal{L}$. We will see that for {claw, net}-free graphs of connectivity one the converse is also true.

4.5 *Let G be a connected graph, $s \in V(G)$, and xsG be a {claw, net}-free graph. Let C be the end-block of vsG distinct from xs, c the boundary vertex of C, $t \in V(C - c)$, and $uv \in E(G)$. Then G has an st-trace containing uv if and only if $(G, s, t, uv) \notin \mathcal{L}$.*

Proof (uses **4.2** and **4.4**). By the above remark, it is sufficient to show that $(G, s, t, uv) \notin \mathcal{L}$ implies that G has an st-trace containing uv. We prove our claim by induction on $v(G)$. If $uv \notin E(C)$ or $V(C) = \{u, v\}$, then our claim follows from **4.4**. Therefore let $uv \in E(C)$. In particular, if $v(C) = 2$, then our claim is true. Therefore let $v(C) \geq 3$, and so C is 2-connected. Let $G' = G - t$ and $C' = C - t$, and so C' is connected.

(**p1**) Suppose that $G - \{u, v\}$ is not connected. Since $(G, s, t, uv) \notin \mathcal{L}$, vertices s and t belong in $G - \{u, v\}$ to different components, say S and T, respectively. Since C is 2-connected, $\bar{T} = T \cup uv$ is also 2-connected.

(**p1**.**1**) Suppose that $v(T) = 1$, i.e. $V(T) = \{t\}$. Then tu is an end-block of $G - v$. Since xsG is {claw, net}-free, by **4.2**, $G - v$ has an st-trace $sPut$. Then $sPuvt$ is an st-trace in G containing uv.

(**p1**.**2**) Now suppose that $v(T) \geq 2$. Since $\bar{T}$ is 2-connected, either $\bar{T} - t$ is 2-connected or t is adjacent in G to an inner vertex z of the end-block of $\bar{T} - t$ avoiding uv. In both cases, $(G', s, z, uv) \notin \mathcal{L}$, and so by the induction hypothesis, G' has a sz-trace P containing uv. Then $sPzt$ is an st-trace containing uv.

(**p2**) Now suppose that $G - \{u, v\}$ is connected. Since $(G, s, t, uv) \notin \mathcal{L}$, $\{u, v\} \neq \{s, t\}$. Since C is 2-connected, t is adjacent to an inner vertex z of the end-block B of xsG' which avoids x. If $t \in \{u, v\}$, say $t = a$, then since $(G, s, t, uv) \notin \mathcal{L}$, v is an inner vertex of B. Then by **4.2**, G' has an sv-trace P, and so $sPba$ is an st-trace containing uv. So let $t \notin \{u, v\}$. Let D be the block of G' containing uv. If $D \neq B$, then since $(G, s, t, uv) \notin \mathcal{L}$, also $(G', s, z, uv) \notin \mathcal{L}$, and so by the induction hypothesis, G' has a sz-trace P containing uv. If $D = B$, then $(G, s, z, uv) \notin \mathcal{L}$ because G has no induced claw centered at z. So again by the induction hypothesis, G' has a sz-trace P containing uv. In both cases $sPzt$ is an st-trace in G containing uv. □

From **4.4** and **4.5** we have:

4.6 *Let G be a {claw, net}-free graph, $v(G) \geq 3$, $\kappa(G) = 1$, $e \in E(G)$, and $\{s, t\} \in V(G)$, $s \neq t$. Then G has an st-trace containing e if and only if s and t are inner vertices of different end-blocks of G and $(G, s, t, e) \notin \mathcal{L}$.*

From **4.6** we have:

4.7 *Let G be a {claw, net}-free graph, $v(G) \geq 3$, $\kappa(G) = 1$, $s \in V(G)$, and $e \in E(G)$. Then G has an s-trace containing e if and only if s is an inner vertex of an end-block in G and $(G, b, s, e) \notin \mathcal{L}$, where b is the boundary vertex of the end-block avoiding s.*

From **4.4** and **4.6** we have the following strengthening of **4.4**.

4.8 *Let G be a connected {claw, net}-free graph having $k \geq 2$ blocks. Let A_j, $j \in \{1, 2\}$, be an end-block of G, a'_j the boundary vertex of A_j, $a_j \in A_j - a'_j$, and $\alpha_j \in E(A_j)$. Let B_i be an inner block of G and $\beta_i \in E(B_i)$. Let $U = \{\alpha_1, \alpha_2\} \cup \{\beta_i : i = 1, \ldots, k-2\}$. Then G has an a_1a_2-trace containing U if and only if*

(c1) $(A_j, a_j, a'_j, \alpha_j) \notin \mathcal{L}$, $j \in \{1, 2\}$ and

(c2) β_i is an inner edge of B_i if $v(B_i) \geq 3$, $i \in \{1, \ldots, k-2\}$.

Let $\mathcal{E}$ denote the set of tuples (G, e) such that G is a 2-connected graph, $e = x_1x_2 \in E(G)$, $G = x_1G_1x_2G_2x_1$, and $G_i \cup x_1x_2$ is 3-connected or a triangle for some $i \in \{1, 2\}$.

Obviously, if e belongs to a track of G, then $(G, e) \notin \mathcal{E}$. The following strengthening of **1.2** shows, in particular, that for 2-connected {claw, net}-free graphs the converse is also true.

4.9 *Let G be a 2-connected {claw, net}-free graph and $e = pz \in E(G)$. Then*

$(a1)$ G has a track,

$(a2)$ the following are equivalent:

$(c1)$ e belongs to a track of G,

$(c2)$ $(G, e) \notin \mathcal{E}$, and

$(a3)$ if $(G, e) \in \mathcal{E}$, then for every inner vertices s, t of the two different blocks S and T of $G - z$ that contain p, there is an st-trace of G containing e.

Proof (uses **3.2** and **4.2** $(a1)$). As we mentioned above, $(c1) \Rightarrow (c2)$.

(**p1**) We prove $(a1)$ and $(c2) \Rightarrow (c1)$ by induction on $v(G)$. The claim holds, if $v(G) = 3$ or G is a cycle. Therefore let $v(G) \geq 4$ and G not a cycle. By $(c2)$, $(G, pz) \notin \mathcal{E}$.

(**p1.1**) Suppose that $G - z$ is 2-connected. Since G is {claw, net}-free, clearly $G - z$ is also {claw, net}-free. Therefore by the induction hypothesis, $G - z$ has a track C, and so $p \in V(C)$. Since G is 2-connected, there is a vertex c in C distinct from p and adjacent to z. Let x and y be the two vertices adjacent to c in C. Then $G' = G - c$ satisfies the assumptions of **3.2**, namely, G' is connected and $P = C - c$ is an xy-trace of $G' - z$. By **3.2**, G' has an st-trace L such that $e \in E(L)$ and $\{s, t\} \subset \{x, y, z\}$. Since c is adjacent to x, y, and z, clearly $csLtc$ is a track of G containing e.

(**p1.2**) Now suppose that $G - z$ is not 2-connected. Let $G - z = AaHbB$, where A and B are end-blocks of G. Since $(G, pz) \notin \mathcal{E}$, p is an inner vertex of an end-block, say $p \in V(A - a)$. Since G is 2-connected, $(G, qz) \notin \mathcal{E}$ for some $q \in V(B - b)$. By **4.2** $(a1)$, $G - z$ has a pq-trace P. Then $zpPqz$ is a track in G containing $e = pz$.

(**p2**) Now we prove $(a3)$. Let $(G, pz) \notin \mathcal{E}$. Then $G - z = SpTbB$, where S is an end-block and T is a block of $G - z$. Let s and t be inner vertices of S and T, respectively. Since G is 2-connected, $G - S$ is connected. Since G is claw-free, $T - S$ is an end-block of $G - S$, and so t and z are inner vertices of different end-blocks of $G - S$. By **4.2** $(a1)$, S has an sp-path P and $G - S$ has a zt-trace Q. Then $sPpzQt$ is an st-trace of G containing e. □

From **4.9** we have, in particular:

4.10 *Let G be a 2-connected {claw, net}-free graph. Then every edge in G belongs to a trace of G.*

In [9] we gave a structural characterization of so-called 'closed' {claw, net}-free graphs. This structure theorem together with the known properties of the Ryjáček closure [10] can be used to provide alternative proofs for some of the above Hamiltonicity results. In [7] we describe some graph closures that are stronger than the closure in [10] and that can be applied to graphs having some induced claws. These results can be used to extend the picture, described in this paper, for a wider class of graphs.

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